ISSN 2347-1921
Volume 12 Number 05
Journal of Advances in Mathematics

Fourier Series Expansions of Powers of the Trigonometric Sine and Cosine Functions

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ABSTRACT

In this paper, Fourier series expansions of powers of sine and cosine functions are established for any possible power–real or complex or positive integer. Recurrence relations are established to facilities the computations of the coefficients of expansions formulae. Numerical applications for real and complex powers are also included , the accuracy of the computed values are at least of order 10^{-10}. While the applications for positive integer powers are given as exact analytical expressions.

Indexing terms/Keywords

Fourier series expansions; fraction and complex powers of sine and cosine; recursive algorithms.

Academic Discipline And Sub-Disciplines

Celestial Mechanics; Dynamical Astronomy.

SUBJECT CLASSIFICATION

Celestial Mechanics.

TYPE (METHOD/APPROACH)

Fourier series expansions of powers of the trigonometric sine and cosine functions powers of sine and cosine functions.

INTRODUCTION

The wide reneges of differentiability, continuity and integralability of the sine and cosine functions, make the Fourier series expansions (FSE) are the most powerful tools for representing periodic functions as sums of these functions. Consequently, (FSE) are the suitable expansions for solving certain classical problems of applied mathematics of these are for examples, the higher order partial differential equations, electromagnetic theory, wave kinematics, rotor-seal systems etc.

On the other hand, the powers of sine and cosine functions appeared in many problems of space dynamics [e.g. in the series of papers by Sharaf, M.A. entitled as "Expansion theory for the elliptic motion of arbitrary eccentricity and semi major axis .Published in Astrophysics and Space Science .since 1981-1986"]., also it appeared in analysis of light curves of eclipsing variable stars.

Due the importance of the (FSE) as mentioned in brief as to in the above, the present paper is devoted to establish, (FSE) of powers of sine and cosine for any possible power–real or complex or positive integer. Recurrence relations are established to facilities the computations of the coefficients of expansions formulae.

Numerical applications for real and complex powers are also included , the accuracy of the computed values are at least of order 10^{-10}. While the applications for positive integer powers are given as exact analytical expressions.

2.FOURIER EXPANSIONS OF $\sin^{2\nu} x$ & $\cos^{2\nu} x$

2.1 Formulations

In this section the (FSE) for any real or complex ν such that $\mathrm{Re}\,\nu \succ -\frac{1}{2}$ (the case of positive integers will be considered in Section 3) will be established for $\sin^{2\nu} x$ $\forall$ $0 \prec x \prec \pi$,and for $\cos^{2\nu} x$ $\forall$ $\frac{-\pi}{2} \prec x \prec \frac{\pi}{2}$ .Since

$$\sin^{p} x = \frac{(-J)^{p}}{2^{p}}(\Phi - \Phi^{-1})^{p}$$

$$\cos^{q} x = \frac{1}{2^{q}}(\Phi + \Phi^{-1})^{q}$$

where

$$J=\sqrt{-1}\ ;\ \Phi=\exp(Jx)\ .$$

Using the binomial theorem, we deduce that

$$\sin^{2\nu}x=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos 2nx;\quad \mathrm{Re}\,\nu\succ-\frac{1}{2}\quad;\quad 0\prec x\prec\pi, \tag{1}$$

$$\cos^{2\nu}x=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(-1)^n a_n\cos 2nx;\quad \mathrm{Re}\,\nu\succ-\frac{1}{2}\quad;\quad -\frac{\pi}{2}\prec x\prec\frac{\pi}{2}, \tag{2}$$

where

$$a_n=\frac{2(-1)^n\Gamma(1+2\nu)}{2^{2\nu}\Gamma(1+\nu+n)\Gamma(1+\nu-n)};n\ge 0 \tag{3}$$

and Γ is the Gamma function.

2.2 Computational developments

2.2.1 Recurrence relations for a's coefficients

■ *For real ν*

Since $\Gamma(x+1)=x\Gamma(x)$,then we get from Equation (3), the recurrence relation

$$a_{n+1}=\frac{n-\nu}{1+n+\nu}a_n;n\ge 0\ , \tag{4.1}$$

where

$$a_0=\frac{\Gamma(1+2\nu)}{2^{2\nu-1}(\Gamma(1+\nu))^2}. \tag{4.2}$$

■ *For complex ν*

Let

$$\nu=\nu_1+J\nu_2\,;J=\sqrt{-1}$$

So, the coefficients becomes complex in he form

$$a_n=q_n+J\,h_n. \tag{5}$$

Consequently, Equation(4.1) becomes

$$q_{n+1}+J\,h_{n+1}=\frac{n-\nu_1-J\nu_2}{1+n+\nu_1+J\nu_2}\left(q_n+J\,h_n\right);n\ge 0.$$

Equating the real and imaginary parts of this equation we get the recurrence relations

$$q_{n+1}=\frac{(n-\nu_1)(1+n+\nu_1)-\nu_2^2}{(1+n+\nu_1)^2+\nu_2^2}q_n+\frac{\nu_2(1+2n)}{(1+n+\nu_1)^2+\nu_2^2}h_n\,;n\ge 0\ , \tag{6.1}$$

$$h_{n+1}=\frac{(n-\nu_1)(1+n+\nu_1)-\nu_2^2}{(1+n+\nu_1)^2+\nu_2^2}h_n+\frac{\nu_2(1+2n)}{(1+n+\nu_1)^2+\nu_2^2}q_n\,;n\ge 0. \tag{6.2}$$

where,

$$q_0=\mathrm{Re}\left[\frac{\Gamma(1+2\nu_1+J\,2\nu_2)}{2^{2\nu_1-1+2J\nu_2}(\Gamma(1+\nu_1+J\nu_2))^2}\right];\ h_0=\mathrm{Im}\left[\frac{\Gamma(1+2\nu_1+J\,2\nu_2)}{2^{2\nu_1-1+2J\nu_2}(\Gamma(1+\nu_1+J\nu_2))^2}\right] \tag{6.3}$$

2.2.2 Accuracy checks

The accuracy of the computed values may be checked by the conditions that:

$$\varepsilon = \sin^{2\nu}(x) - \left(\frac{1}{2}a_0 + \sum_{n=1} a_n \cos 2nx\right)_{\forall \in x]0,\pi[} ,$$

$$\varepsilon_1 = \cos^{2\nu}(x) - \left(\frac{1}{2}a_0 + \sum_{n=1} (-1)^n a_n \cos 2nx\right)_{\forall x \in]-\pi/2,\pi/2[} ,$$

where ε and ε_1 are small tolerances of $\approx O(10^{-10})$ at least.

2.2.3 Numerical examples

Example 1

Consider $\nu = 5.94$, with this value of ν we get for the coefficients of (FSE) of $\sin^{2\nu} x$ the first twenty values which are listed in columns 2 & 6 from table 1, where $a_0 = 0.453349713$. The values of angle x are listed in columns 3 & 7,while, the accuracy cheeks ε are given in columns 4 & 8.

Example 2

The corresponding computations for $\cos^{2\nu} x$ are listed in Table 2, where $\nu = 3.765$ and $a_0 = 0.562594$.

Example 3

The results for Fourier expansion of $\sin^{2\nu} x$ with complex $\nu = 2.67 + 2i$ and $a_0 = 0.572892 - 0.176663i$ are listed in Table 3.

Example 4

Finally, the results for Fourier expansion of $\cos^{2\nu} x$ with complex $\nu = 3.765 + 2i$ and $a_0 = 0.518449 - 0.121644i$ are listed in Table 4.

Table 1. Fourier Expansion of $\sin^{2\nu} x$ and its Error Analysis

i	a_i	x^0	ε	i	a_i	x^0	ε
1	-0.388026	1.61731	8.78464×10^{-14}	11	9.82691×10^{-10}	51.9953	6.80289×10^{-14}
2	0.241416	87.4289	2.44527×10^{-14}	12	2.77169×10^{-10}	10.5793	-7.18869×10^{-14}
3	-0.106396	35.874	-1.86795×10^{-14}	13	8.86824×10^{-11}	24.997	1.63869×10^{-13}
4	0.0314693	38.3996	-1.14825×10^{-13}	14	3.13991×10^{-11}	164.199	1.84519×10^{-13}
5	-0.00558047	161.314	-1.78468×10^{-14}	15	1.20858×10^{-11}	28.1381	-1.06887×10^{-13}
6	0.000439334	37.2613	-1.16629×10^{-13}	16	4.99076×10^{-12}	32.1528	7.56062×10^{-14}
7	2.0371×10^{-6}	80.7445	-7.79654×10^{-14}	17	2.18863×10^{-12}	86.5704	6.71407×10^{-14}
8	1.54901×10^{-7}	89.0032	-6.22002×10^{-14}	18	1.01112×10^{-12}	135.915	1.95399×10^{-14}
9	2.13585×10^{-8}	160.248	-1.44357×10^{-13}	19	4.88938×10^{-13}	32.1719	7.71883×10^{-14}
10	4.10019×10^{-9}	144.703	3.60545×10^{-14}	20	2.46165×10^{-13}	51.9078	7.28029×10^{-14}

Table 2. Fourier Expansion of $\cos^{2\nu} x$ and its Error Analysis

1	0.444526	88.3827	1.48426×10^{-10}	11	-5.65227×10^{-8}	38.0047	8.79626×10^{-11}
2	0.213203	2.57109	3.19526×10^{-11}	12	2.59398×10^{-8}	79.4207	-2.07767×10^{-10}
3	0.055625	54.126	-4.97047×10^{-11}	13	-1.27417×10^{-8}	65.003	2.43607×10^{-10}
4	0.00548012	51.6004	-1.5588×10^{-10}	14	6.62367×10^{-9}	-74.1991	3.27874×10^{-10}
5	-0.000146928	-71.3137	-8.5577×10^{-11}	15	-3.61275×10^{-9}	61.8619	-1.82286×10^{-10}
6	0.0000185824	52.7387	-1.75916×10^{-10}	16	2.05359×10^{-9}	57.8472	1.33713×10^{-10}
7	-3.85802×10^{-6}	9.25552	-1.09553×10^{-10}	17	-1.21×10^{-9}	3.42959	9.08718×10^{-11}
8	1.06083×10^{-6}	0.996754	-8.76215×10^{-11}	18	7.35786×10^{-10}	-45.9153	1.24942×10^{-11}
9	-3.51949×10^{-7}	-70.2482	-2.5846×10^{-10}	19	-4.60089×10^{-10}	57.8281	1.35819×10^{-10}
10	1.3385×10^{-7}	-54.7029	2.92011×10^{-11}	20	2.94949×10^{-10}	38.0922	9.51788×10^{-11}

ISSN 2347-1921
Volume 12 Number 05
Journal of Advances in Mathematics

Table 3. Fourier Expansion of $\sin^{2\nu} x$ with complex $\nu = 2.67 + 2i$ and its Error Analysis

i	a_i	x^0	□
1	□0.47276 □0.0739583 □	1.61731	3.34711 □ $10^{□10}$ □5.36575 □ $10^{□10}$ □
2	0.233323 □0.0760953 □	87.4289	6.31528 □ $10^{□12}$ □2.28166 □ $10^{□11}$ □
3	□0.0292874 □0.080962 □	35.874	□2.82208 □ $10^{□12}$ □5.69501 □ $10^{□12}$ □
4	□0.0222893 □0.0114597 □	38.3996	□3.59518 □ $10^{□12}$ □1.64443 □ $10^{□11}$ □
5	0.00108319 □0.00751676 □	161.314	□6.83748 □ $10^{□12}$ □3.88986 □ $10^{□11}$ □
6	0.00231047 □0.00123722 □	37.2613	1.05687 □ $10^{□11}$ □3.9168 □ $10^{□11}$ □
7	0.00099812 □0.000258246 □	80.7445	□3.48033 □ $10^{□12}$ □1.2284 □ $10^{□11}$ □
8	0.000291682 □0.000346561 □	89.0032	6.44379 □ $10^{□12}$ □2.32346 □ $10^{□11}$ □
9	0.0000370438 □0.000214621 □	160.248	□5.81363 □ $10^{□12}$ □5.35141 □ $10^{□12}$ □
10	□0.0000324126 □0.000107957 □	144.703	1.17347 □ $10^{□11}$ □4.06876 □ $10^{□11}$ □

Table 4. Fourier Expansion of $\cos^{2\nu} x$ with complex $\nu = 3.76 + 2i$ and its Error Analysis

i	a_i	x^0	□
1	0.435053 □0.0611116 □	88.3827	□1.87976 □ $10^{□11}$ □8.81187 □ $10^{□11}$ □
2	0.242827 □0.0373769 □	2.57109	□2.68618 □ $10^{□13}$ □1.34417 □ $10^{□12}$ □
3	0.0702689 □0.0607666 □	54.126	6.62803 □ $10^{□14}$ □3.85721 □ $10^{□12}$ □
4	□0.00236789 □0.0246954 □	51.6004	□2.45787 □ $10^{□12}$ □1.22795 □ $10^{□11}$ □
5	□0.00555658 □0.0000654824 □	□71.3137	3.18412 □ $10^{□12}$ □2.2592 □ $10^{□11}$ □
6	0.000436255 □0.00123569 □	52.7387	1.56281 □ $10^{□12}$ □4.79623 □ $10^{□12}$ □
7	0.000194994 □0.000301375 □	9.25552	□1.26582 □ $10^{□12}$ □6.31466 □ $10^{□12}$ □
8	□0.000110119 □0.0000310005 □	0.996754	□3.04423 □ $10^{□13}$ □1.42518 □ $10^{□12}$ □
9	0.0000393337 □0.0000131311 □	□70.2482	2.15089 □ $10^{□12}$ □1.06969 □ $10^{□12}$ □
10	□0.0000112576 □0.0000123446 □	□54.7029	□2.2678 □ $10^{□12}$ □8.69305 □ $10^{□12}$ □

3.TRIGONOMETRIC SERIES REPRESENTATIONS OF $\sin^n\varphi$ & $\cos^n\varphi$

3.1 Formulations

For n positive integer we deduce from Equations (1) & (2) after some analysis that:

$$\sin^n\varphi = \begin{cases} \frac{1}{2}A_0^{(n)} + \sum_{s=1}^{(n+\delta)/2} A_s^{(n)} \cos 2s\varphi & ; \quad \text{if } n \equiv \text{even} \;;\; \delta = 0 \\ \sum_{s=1}^{(n+\delta)/2} A_s^{(n)} \sin(2s-1)\varphi & ; \quad \text{if } n \equiv \text{odd} \;;\; \delta = 1 \end{cases}$$

$$\cos^n\varphi = \begin{cases} \frac{1}{2}C_0^{(n)} + \sum_{s=1}^{(n+\delta)/2} C_s^{(n)} \cos 2s\varphi & ; \quad \text{if } n \equiv \text{even} \;;\; \delta = 0 \\ \sum_{s=1}^{(n+\delta)/2} C_s^{(n)} \cos(2s-1)\varphi & ; \quad \text{if } n \equiv \text{odd} \;;\; \delta = 1 \end{cases}$$

$$C_s^{(m)} = 2^{-m+1}\binom{m}{\frac{m+\delta}{2} - s} \qquad ; \; A_s^{(m)} = (-1)^{s+\delta}\, 2^{-m+1}\binom{m}{\frac{m+\delta}{2} - s}$$

We can check the accuracy of the computed values from the conditions that:

- If n even positive integer, then:

$$\frac{1}{2}A_0^{(n)}+\sum_{s=1}^{n/2}A_s^{(n)}=0\ ;\ \frac{1}{2}C_0^{(n)}+\sum_{s=1}^{n/2}C_s^{(n)}=1$$

- If n odd positive integer, then:

$$\sum_{s=1}^{(n+1)/2}(2s-1)A_s^{(n)}=0\ ;\ \sum_{s=1}^{(n+1)/2}C_s^{(n)}=1$$

3.2.Applications

Tables 5 & 6 give the trigonometric series representations of $\sin^n\varphi$ & $\cos^n\varphi$; respectively ; $n=2,3,\ldots,10$.

Table 5. Trigonometric series representations of $\sin^n\varphi, n=2,3,\ldots,10$

$\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos(2\theta)$
$\sin^3\theta=\frac{3\sin(\theta)}{4}-\frac{1}{4}\sin(3\theta)$
$\sin^4\theta=-\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)+\frac{3}{8}$
$\sin^5\theta=\frac{5\sin(\theta)}{8}-\frac{5}{16}\sin(3\theta)+\frac{1}{16}\sin(5\theta)$
$\sin^6\theta=-\frac{15}{32}\cos(2\theta)+\frac{3}{16}\cos(4\theta)-\frac{1}{32}\cos(6\theta)+\frac{5}{16}$
$\sin^7\theta=\frac{35\sin(\theta)}{64}-\frac{21}{64}\sin(3\theta)+\frac{7}{64}\sin(5\theta)-\frac{1}{64}\sin(7\theta)$
$\sin^8\theta=-\frac{7}{16}\cos(2\theta)+\frac{7}{32}\cos(4\theta)-\frac{1}{16}\cos(6\theta)+\frac{1}{128}\cos(8\theta)+\frac{35}{128}$
$\sin^9\theta=\frac{63\sin(\theta)}{128}-\frac{21}{64}\sin(3\theta)+\frac{9}{64}\sin(5\theta)-\frac{9}{256}\sin(7\theta)+\frac{1}{256}\sin(9\theta)$
$\sin^{10}\theta=-\frac{105}{256}\cos(2\theta)+\frac{15}{64}\cos(4\theta)-\frac{45}{512}\cos(6\theta)+\frac{5}{256}\cos(8\theta)-\frac{1}{512}\cos(10\theta)+\frac{63}{256}$

Table 6. Trigonometric series representations of $\cos^n\varphi, n = 2,3,...,10$

$$\cos^2\theta = \frac{1}{2}\cos(2\theta) + \frac{1}{2}$$

$$\cos^3\theta = \frac{3\cos(\theta)}{4} + \frac{1}{4}\cos(3\theta)$$

$$\cos^4\theta = \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta) + \frac{3}{8}$$

$$\cos^5\theta = \frac{5\cos(\theta)}{8} + \frac{5}{16}\cos(3\theta) + \frac{1}{16}\cos(5\theta)$$

$$\cos^6\theta = \frac{15}{32}\cos(2\theta) + \frac{3}{16}\cos(4\theta) + \frac{1}{32}\cos(6\theta) + \frac{5}{16}$$

$$\cos^7\theta = \frac{35\cos(\theta)}{64} + \frac{21}{64}\cos(3\theta) + \frac{7}{64}\cos(5\theta) + \frac{1}{64}\cos(7\theta)$$

$$\cos^8\theta = \frac{7}{16}\cos(2\theta) + \frac{7}{32}\cos(4\theta) + \frac{1}{16}\cos(6\theta) + \frac{1}{128}\cos(8\theta) + \frac{35}{128}$$

$$\cos^9\theta = \frac{63\cos(\theta)}{128} + \frac{21}{64}\cos(3\theta) + \frac{9}{64}\cos(5\theta) + \frac{9}{256}\cos(7\theta) + \frac{1}{256}\cos(9\theta)$$

$$\cos^{10}\theta = \frac{105}{256}\cos(2\theta) + \frac{15}{64}\cos(4\theta) + \frac{45}{512}\cos(6\theta) + \frac{5}{256}\cos(8\theta) + \frac{1}{512}\cos(10\theta) + \frac{63}{256}$$

In concluded the present paper, we stress that, Fourier series expansions of powers of sine and cosine functions are established for any possible power–real or complex or positive integer.

Recurrence relations are established to facilities the computations of the coefficients of expansions formulae. Numerical applications for real and complex powers are also included , the accuracy of the computed values are at least of order 10^{-10}. While the applications for positive integer powers are given as exact analytical expressions.

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